State Estimation in Navigation

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Navigation

- Navigation – when a vehicle determines position, velocity and time (PVT) on board
- Observations can be angle, range, Doppler shift, and output of Inertial Measurement Units – erroneous, noisy
- State estimation algorithms are required to estimate the PVT from erroneous and noisy observations which are often a non-linear function of PVT of the user (vehicle)

Posing navigation as an estimation problem

Vehicle dynamics:

$$\dot{Y}(t) = f(Y, t) + \omega(t)$$

Where

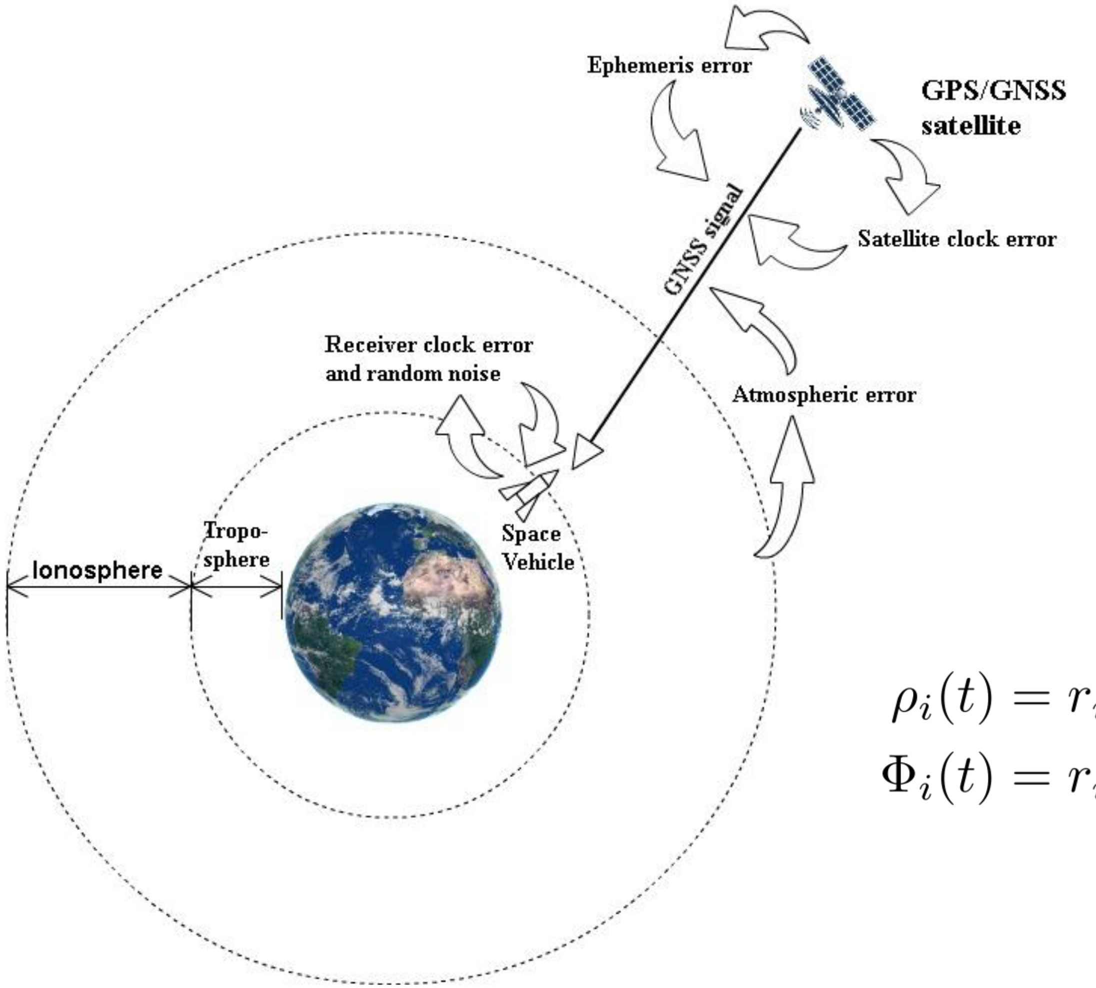
$$Y = \begin{bmatrix} r \\ v \end{bmatrix}$$

$\omega(t)$ is zero mean white noise

Observation/measurement:

$$Z_k = h(Y_k) + \nu_k$$

LEO satellite navigation using pseudo-range and carrier range observables



$$\rho_i(t) = r_i(t) + c[\delta t_u(t) - \delta t_i(t-\tau)] + I_\rho(t) + \epsilon_\rho(t)$$
$$\Phi_i(t) = r_i(t) + c[\delta t_u(t) - \delta t_i(t-\tau)] + I_\Phi(t) + N\lambda + \epsilon_\Phi(t)$$

Non-Bayesian approach

- Does not consider any prior information regarding the state to be estimated
- Example – Maximum Likelihood Estimator (MLE), Least Squares Estimator (LSE)
- MLE: $$\widehat{Y}^{ML} = \arg\max_{Y} \Lambda(Y) = \arg\max_{Y} p(Z|Y)$$
- LSE: $$\widehat{Y}^{LS} = \arg\min_{Y} |Z - h(Y)|^2$$
- Both are philosophically different, but coincides under Gaussian noise assumption

LSE – commonly used for PVT estimation using GNSS

- State to be estimated – $Y = [x_u \ y_u \ z_u \ \Delta b]^T$
- Linearising the cost function mentioned earlier, one can write

$$\Delta\hat{Y} = (H^T H)^{-1} H^T \Delta\rho$$

Here

$$H = \frac{\partial Z}{\partial Y}\bigg|_{Y_0} = \begin{bmatrix} -\frac{x_1 - x_0}{R_{1,0}} & -\frac{y_1 - y_0}{R_{1,0}} & -\frac{z_1 - z_0}{R_{1,0}} & 1 \\ -\frac{x_2 - x_0}{R_{2,0}} & -\frac{y_2 - y_0}{R_{2,0}} & -\frac{z_2 - z_0}{R_{2,0}} & 1 \\ -\frac{x_3 - x_0}{R_{3,0}} & -\frac{y_3 - y_0}{R_{3,0}} & -\frac{z_3 - z_0}{R_{3,0}} & 1 \\ -\frac{x_4 - x_0}{R_{4,0}} & -\frac{y_4 - y_0}{R_{4,0}} & -\frac{z_4 - z_0}{R_{4,0}} & 1 \end{bmatrix}$$

This is solved iteratively

Bayesian Approach

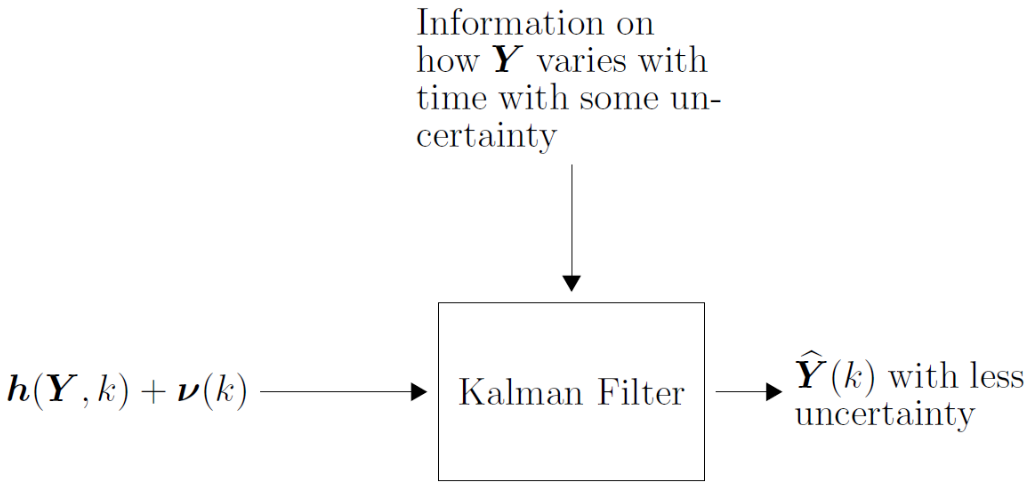
- Considers prior state information
- Based on Bayes' rule

$$p(Y|Z) = \frac{p(Z|Y)p(Y)}{p(Z)}$$

- Sequential estimators like Kalman Filter, Particle Filter are Bayesian estimators

Kalman Filter

- Kalman Filter is a class of algorithms used mostly for estimation of some quantity on the fly
- It is a Filter in the sense that it reduces the effect of noise of the measurement(s) on the quantity to be estimated
- It is not a "Filter" like analog or digital filters



Kalman Filter Frame-work

A liner discrete estimation problem:

$$Y_{k+1} = \Phi_k Y_k + \omega_k$$

$$Z_k = H_k Y_k + \nu_k$$

Terminologies:

$$\begin{aligned}
\text{Error } &= \Delta Y_k \\
P_k &= E[\Delta Y_k \Delta Y_k^T] \rightarrow \text{Error covariance} \\
Q_k &= E[\omega_k \omega_k^T] \rightarrow \text{Process noise covariance} \\
R_k &= E[\nu_k \nu_k^T] \rightarrow \text{Measurement noise covariance}
\end{aligned}$$

Kalman Filter Frame-work

Prediction

$$\text{State Prediction:} \quad \hat{Y}_k^- = \Phi_{k-1}\hat{Y}_{k-1}^+$$

$$\text{Covariance Prediction:} \quad P_k^- = \Phi_{k-1}P_{k-1}^+\Phi_{k-1}^T + Q_{k-1}$$

Update:

$$\text{Kalman Gain:} \quad K_k = P_k^- H_k^T (H_k P_k^- H_k^T + R_k)^{-1}$$

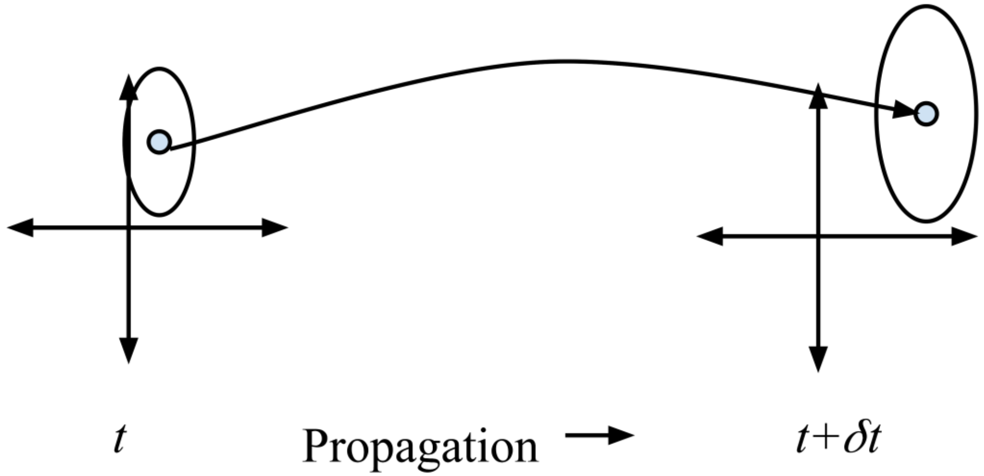
$$\text{State Update:} \quad \hat{Y}_k^+ = \hat{Y}_k^- + K_k(Z_k - H_k\hat{Y}_k^-)$$

$$\text{Covariance Update (Joseph Form):} \quad P_k^+ = (I - K_kH_k)P_k^-(I - K_kH_k)^T + K_kR_kK_k^T$$

How to accommodate nonlinearity

$$\dot{\boldsymbol{Y}}(t) = \boldsymbol{f}(t, \boldsymbol{Y}(t)) + \boldsymbol{\nu}(t)$$
$$\boldsymbol{Z}(k) = \boldsymbol{h}(\boldsymbol{Y}(k)) + \boldsymbol{\omega}(k)$$

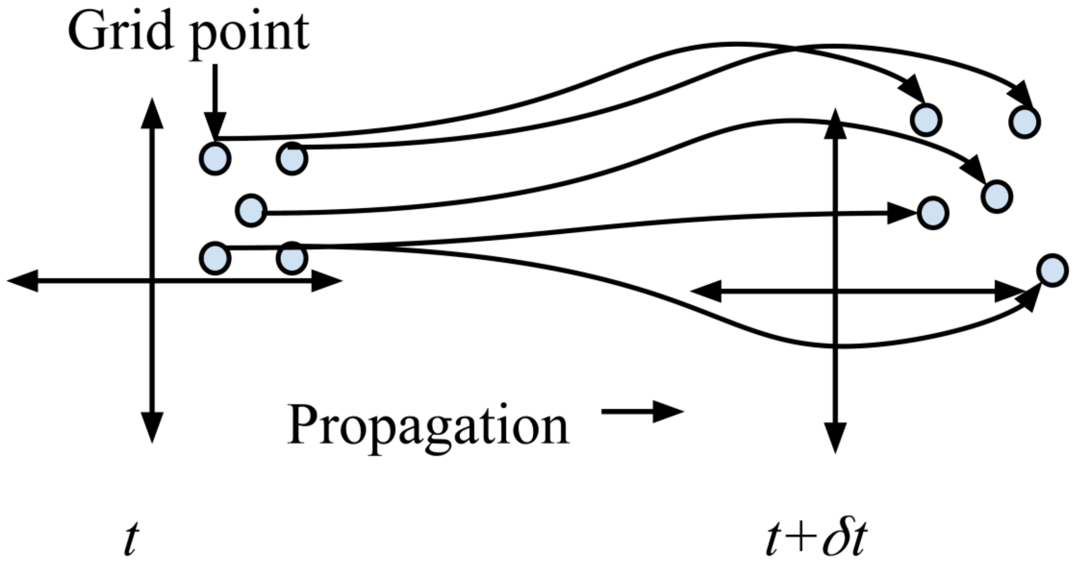
The EKF propagates the state using non-linear the function and error covariance using linearization



$$\widehat{\boldsymbol{Y}}^{-}(t+\delta t) = \widehat{\boldsymbol{Y}}^{+}(t) + \int_{t}^{t+\delta t} \boldsymbol{f}(\tau, \boldsymbol{Y}(\tau)) d\tau$$
$$= \boldsymbol{F}(t, \widehat{\boldsymbol{Y}}^{+}(t))$$

A More Accurate approach- The Unscented Kalman Filter

The UKF uses deterministic sampling to predict the *a priori* mean and error covariance

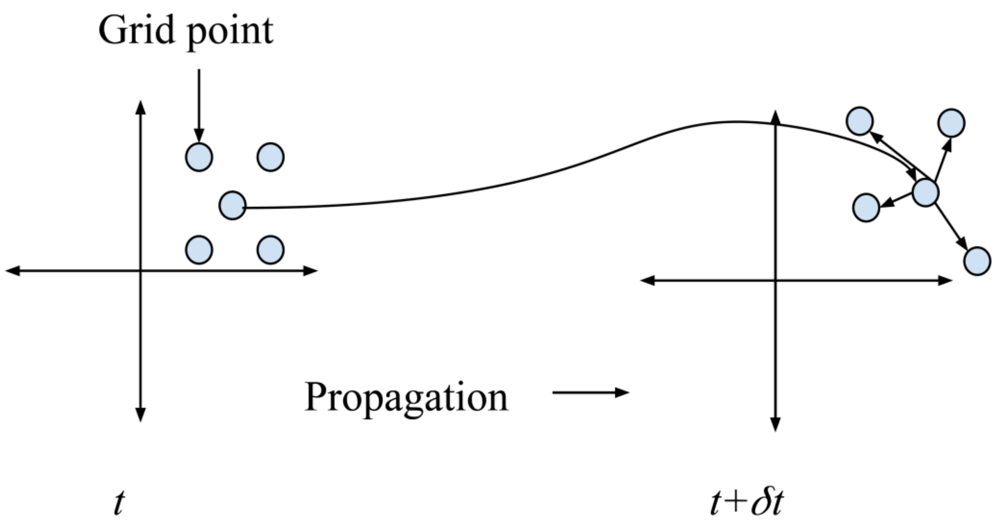


Single Propagation Unscented Kalman Filter (SPUKF)

- Only one sigma point is propagated to the next time step
- Other 2n sigma points are calculated from previous information

$$\boldsymbol{Y}_i(t+\delta t) \approx \boldsymbol{F}(t, \widehat{\mathbf{Y}}^{+}(t)) + \boldsymbol{D}_{\boldsymbol{\Delta Y_i}}\boldsymbol{F}$$
$$= \mathbf{Y}_0(t+\delta t) + \left.\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Y}}\right|_{\widehat{\mathbf{Y}}^{+}(t)} \Delta \boldsymbol{Y}_i$$

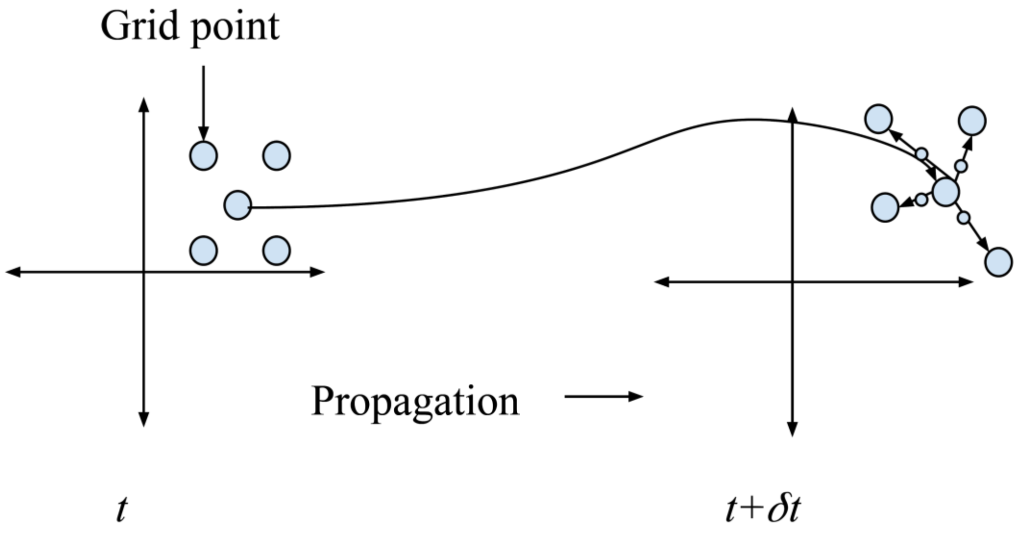
$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{Y}} = e^{\mathcal{J}\delta t}$$



S. K. Biswas, L. Qiao, and A. G. Dempster, "A Novel a Priori State Computation Strategy for the Unscented Kalman Filter to Improve Computational Efficiency," ***IEEE Transactions on Automatic Control***, vol. 62, no. 4, pp. 1852–1864, Apr. 2017, doi: 10.1109/TAC.2016.2599291.

Extrapolated Single Propagation Unscented Kalman Filter (ESPUKF)

- Estimation error for the SPUKF is higher than the UKF due to the 2nd order Taylor series terms
- ESPUKF reduces the error using Richardson Extrapolation



Extrapolated Single Propagation Unscented Kalman Filter (ESPUKF)

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$$N_1(\boldsymbol{\Delta Y_i}) = \boldsymbol{F}(t, \widehat{\boldsymbol{Y}}^+(t)) + \boldsymbol{D}_{\boldsymbol{\Delta Y_i}} \boldsymbol{F}|_{\widehat{\boldsymbol{Y}}^+(t)}$$

$$\begin{aligned} N_2(\frac{\boldsymbol{\Delta Y_i}}{2}) =& \boldsymbol{F}(t, \widehat{\boldsymbol{Y}}^+(t)) + \boldsymbol{D}_{\boldsymbol{\Delta Y_i/2}} \boldsymbol{F}|_{\widehat{\boldsymbol{Y}}^+(t)} \\ &+ \boldsymbol{D}_{\boldsymbol{\Delta Y_i/2}} \boldsymbol{F}|_{\widehat{\boldsymbol{Y}}^+(t)+\frac{\boldsymbol{\Delta Y_i}}{2}} \end{aligned}$$

$$\begin{aligned} \boldsymbol{Y}_i(t+\delta t) =& 2N_2(\frac{\boldsymbol{\Delta Y_i}}{2}) - N_1(\boldsymbol{\Delta Y_i}) \\ &- \left[\frac{1}{2}\frac{\boldsymbol{D}^3_{\boldsymbol{\Delta Y_i}}}{3!} + \frac{3}{4}\frac{\boldsymbol{D}^4_{\boldsymbol{\Delta Y_i}}}{4!} + \ldots\right] \boldsymbol{F}|_{\widehat{\boldsymbol{Y}}^+(t)} \end{aligned}$$

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Implementation for LEO satellite navigation

- Requirements:
  - System model
  - State transition matrix
  - Measurement model
  - Measurement Jacobian
  - Initial mean state and error covariance
  - Process and measurement noise matrices

Satellite Dynamics - Newton's Law of gravitation

The gravitational force between two masses M and m at a distance r between their centre of mass is:

$$F = -\frac{GMm}{r^2}$$

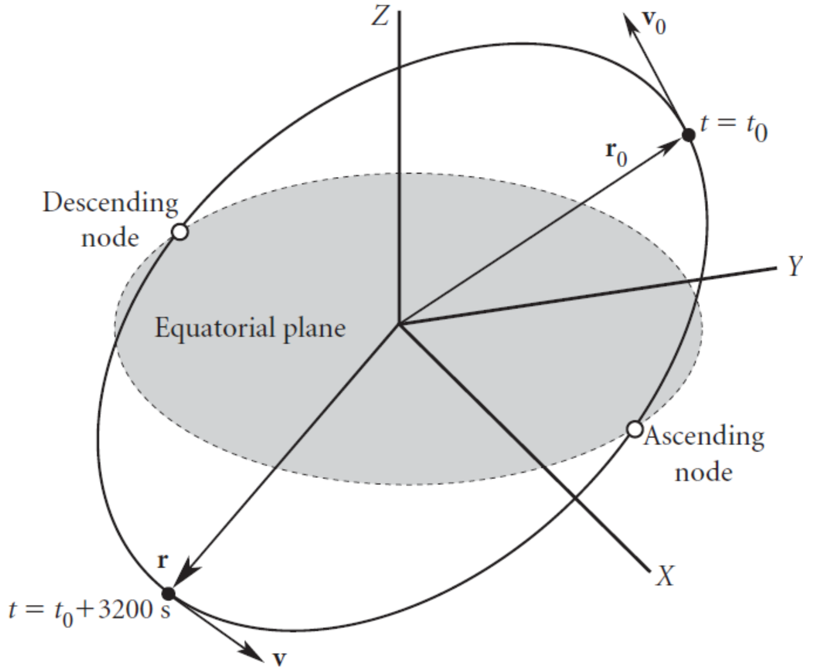
Or acceleration of m:

$$a = -\frac{GM}{r^2}$$

Equation of motion of a satellite in 3 dimension:

$$\ddot{r} = -\frac{\mu}{r^3} r$$

Here, $\mu = GM$



Satellite Dynamics

In reality the acceleration is:

$$\ddot{r} = -\frac{\mu}{r^3} r + a_p$$

Where a_p is acceleration due to perturbation forces

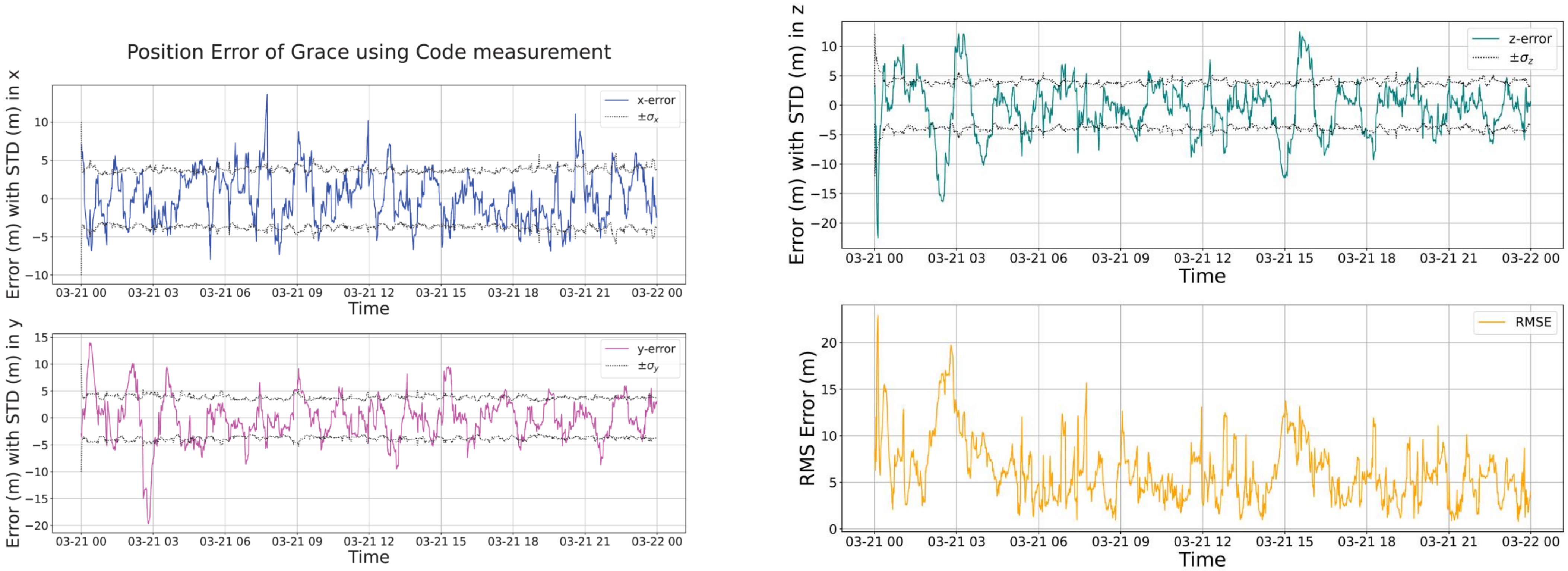
State space model

$$\begin{bmatrix} \dot{r} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ -\frac{\mu}{r^3} & 0_{3\times3} \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix} + \begin{bmatrix} 0_{3\times1} \\ a_p \end{bmatrix}$$

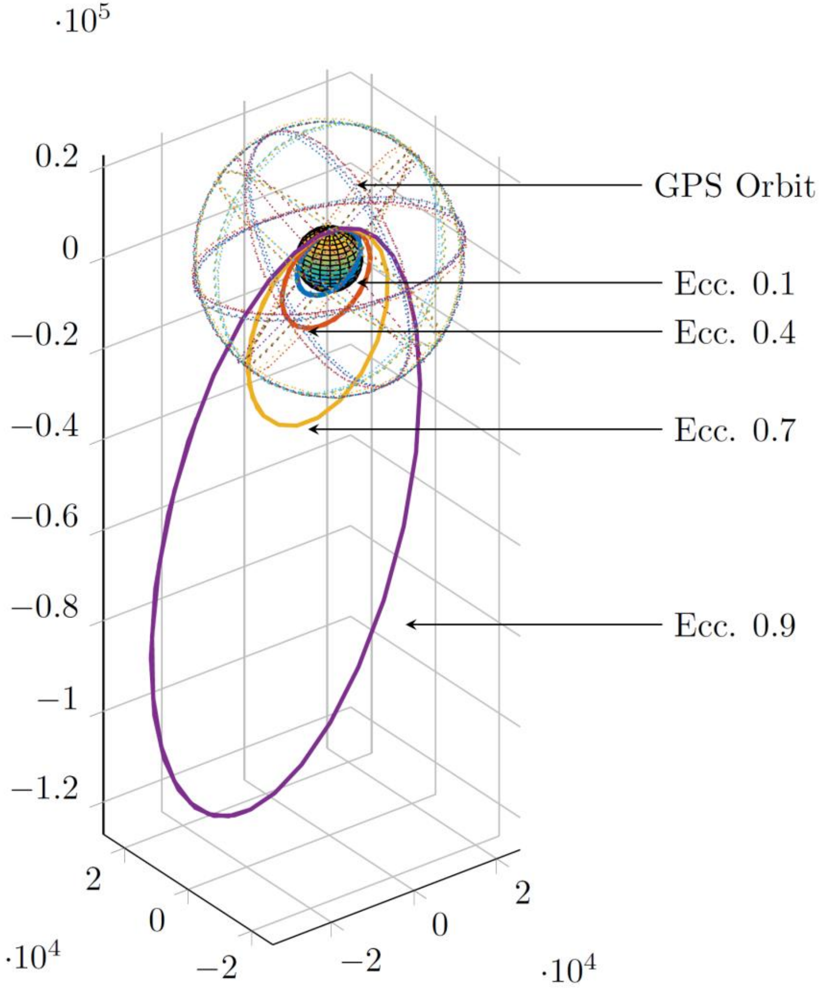
Sources of the perturbation forces:

- Non-uniform gravitational field of the earth
- Gravitational forces from other bodies
- Atmospheric drag
- Solar radiation pressure

GRACE Orbit Determination



Navigation performance in elliptical orbit



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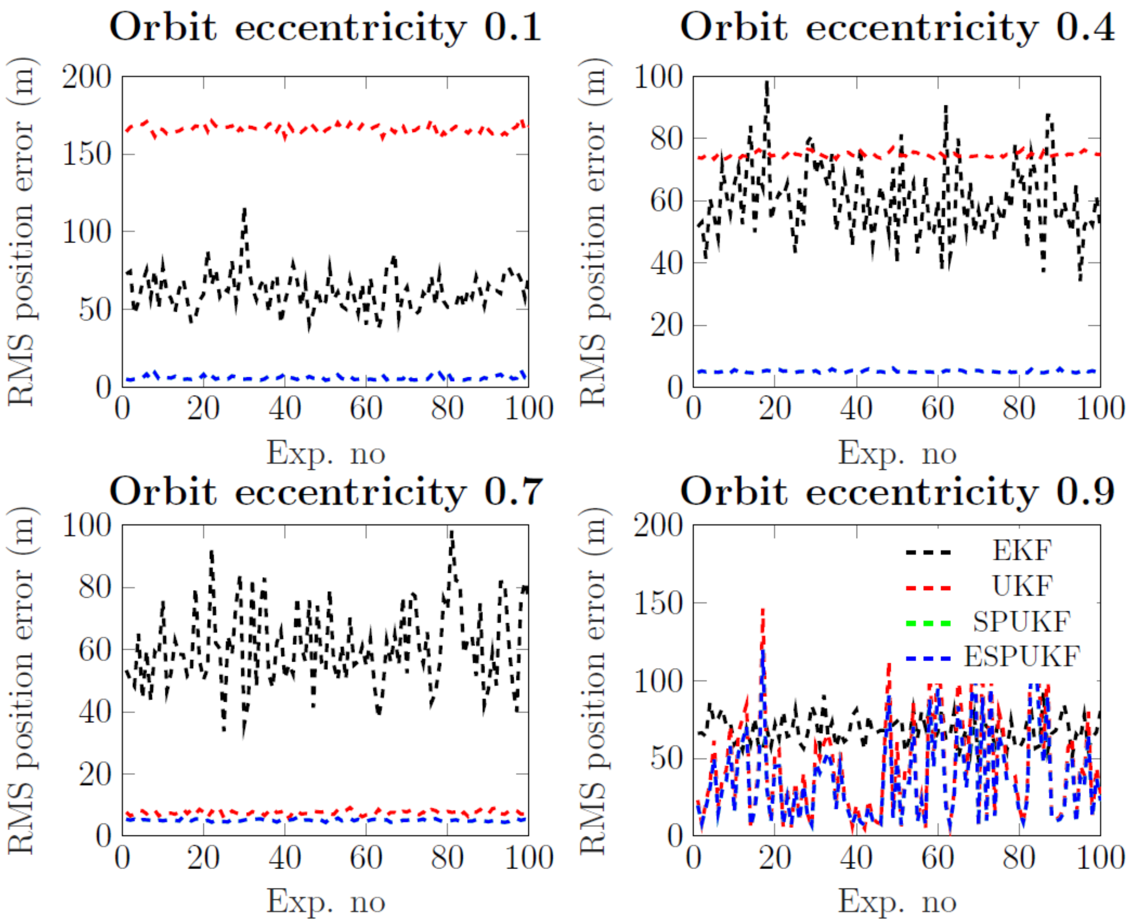

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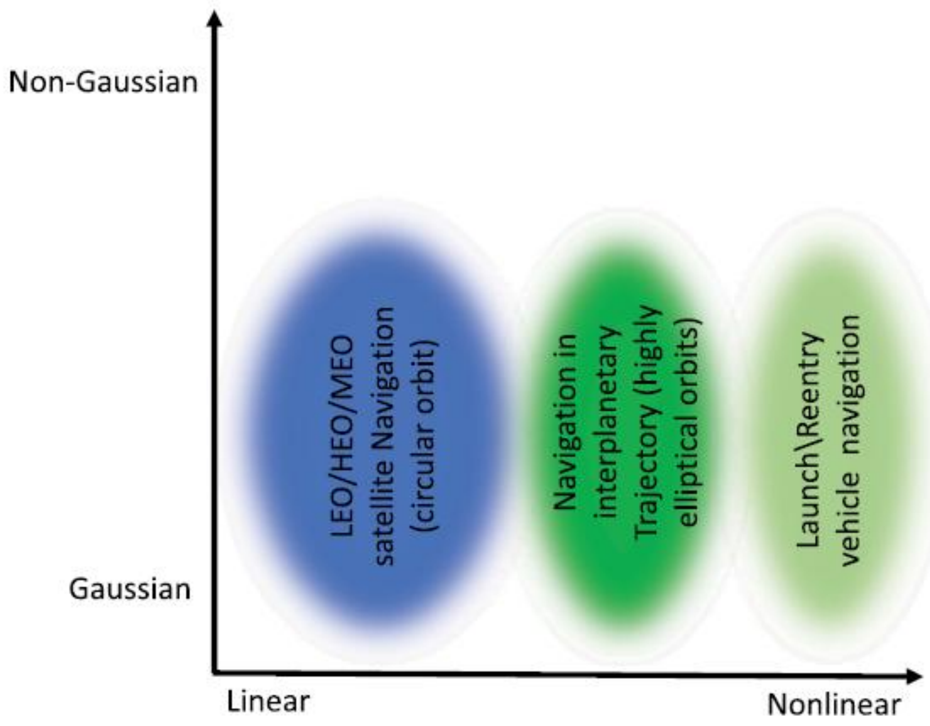
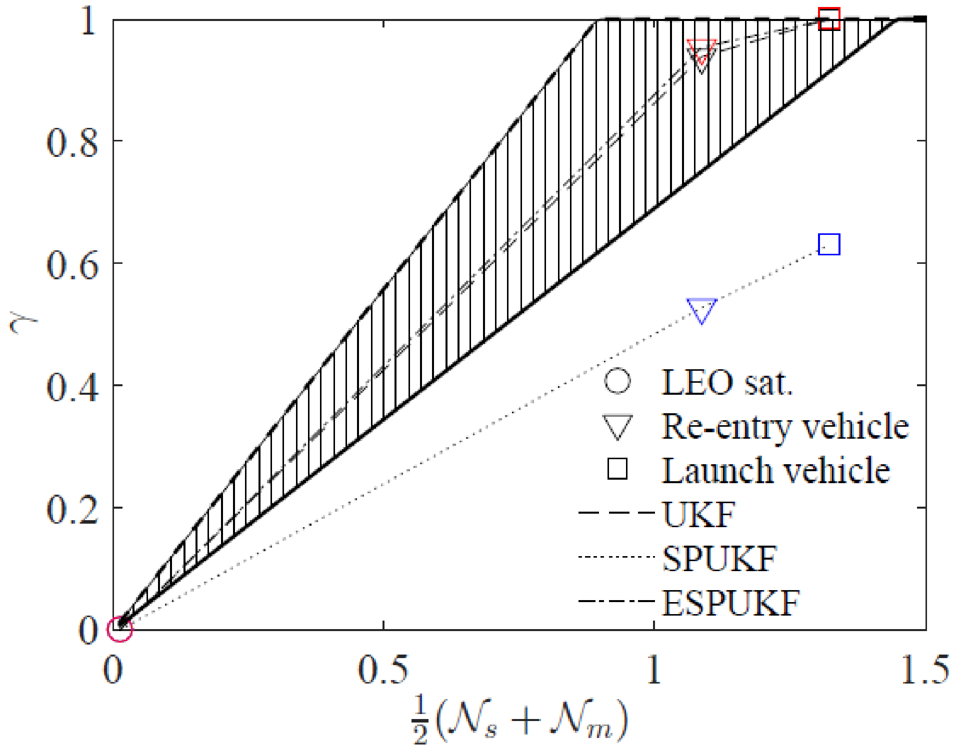
Non-linearity and Estimation performance





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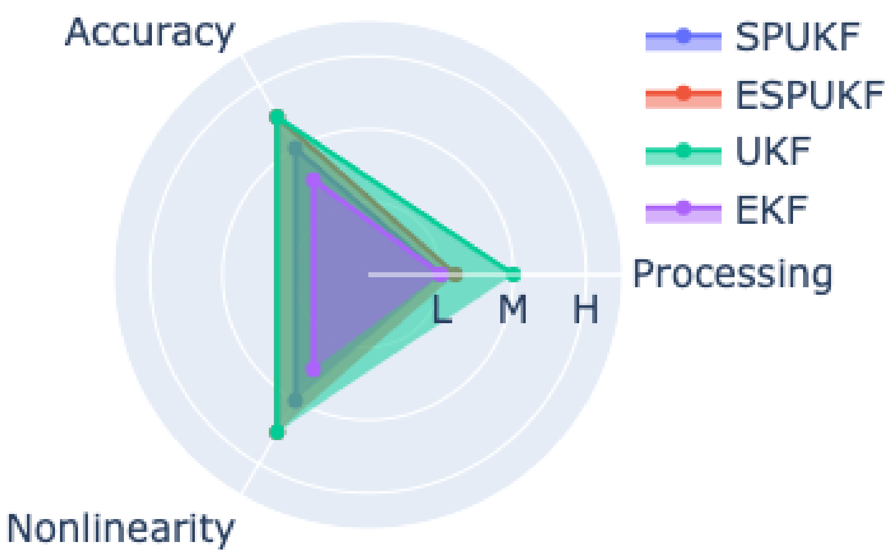
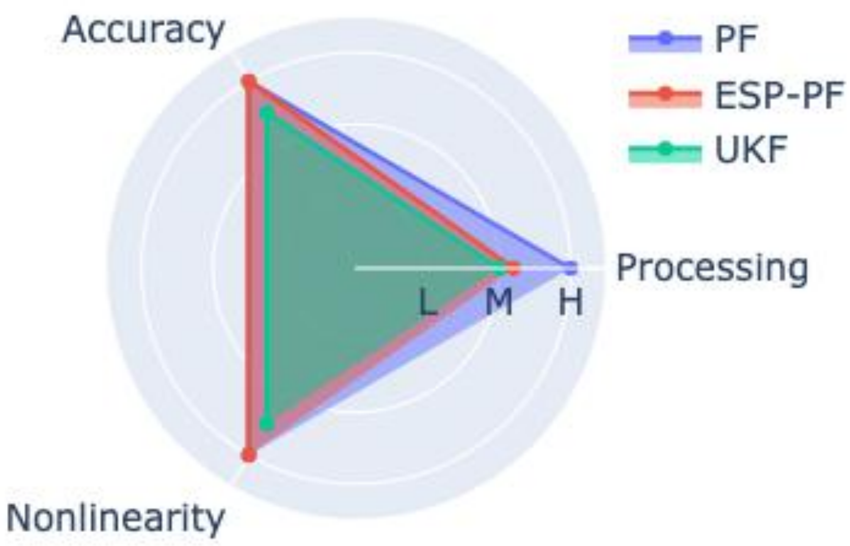
S. K. Biswas, 'Computationally Efficient Non-linear Kalman Filters for On-board Space Vehicle Navigation', UNSW Sydney, Sydney, Australia, 2017.

Choosing the right algorithm

Considerations:

- Computational resources are limited
- Observations are not highly accurate
- Nonlinearity in system and measurement





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